

# Bohmian trajectories for bipartite entangled states

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We derive Bohm's trajectories from Bell's beables for arbitrary bipartite systems composed by dissipative noninteracting harmonic oscillators at finite temperature. As an application of our result, we calculate the Bohmian trajectories of particles described by a generalized Werner state, comparing the trajectories when the state is either separable or entangled. We show that qualitative differences appear in the trajectories for entangled states as compared with those for separable states.

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## I. INTRODUCTION

Entanglement phenomenon is possibly the most striking feature of Quantum Mechanics, playing a key role in quantum information processing and quantum computing. The striking feature of entanglement was experimentally turned possible after the seminal paper by Bell [1], and since then efforts from experimental and theoretical research to demonstrate the violation of Bell's inequalities have been undertaken. To unequivocally demonstrate violation of Bell's inequality, a major development in experimental techniques has been carried out to produce entangled photons from the cascade atomic process [2] to the parametric down-conversion processes [3]. Massive entangled particles have also been produced through radiation-matter interaction in cavity QED [4] and trapped ions [5]. In the latter case, a controlled entanglement of 14 quantum bits has been recently generated enabling the implementation of the largest quantum register to date [6]. The violation of a form of Bell's inequality has also been verified with massive entangled particles within an ion trap [7]. Parallel to the experimental achievements, theoretical physics has been struggling with recently advanced striking features of entanglement, such as the derivation of separability criterion for density matrices [8], entanglement sudden death [9], and quantum discord [10].

Bohmian mechanics is a theory equivalent with orthodox quantum mechanics having the advantage of providing ontological meaning for the quantum particle trajectory [11]. Bohmian mechanics assumes that the complete description of particle systems is provided by its wave function  $\Psi$  and its configuration  $Q = (Q_1, \dots, Q_N) \in \mathbb{R}^{3N}$ , where  $Q_\alpha$  is the position of the  $\alpha$ -th particle. While the wave function  $\Psi(Q)$  evolves according to the Schrödinger's equation, the motion of the particles evolves according to the equation  $m_\alpha dQ_\alpha/dt = \hbar \text{Im} \{ \Psi^{-1} \partial \Psi / \partial Q_\alpha \}$ . Since only the average trajectories are experimentally accessible, the particle positions are the "hidden variables" of Bohmian mechanics.

In the present work we focus on entangled states of bipartite dissipative systems from the Bohmian trajectories perspective as formulated by Vink's extension of Bell's beables [12, 13]. Our goal is to verify what happens to the Bohmian trajectories of a given bipartite state at the instant occurring separability. To achieve our goal, we first derive Bohm's trajectories from Bell's beables for arbitrary bipartite states under thermal reservoirs at finite temperature. On this regard, we note that an extension of Bell's beables that encompasses dissipation and decoherence for one particle state has been advanced [14], where the diffusive terms in Nelson's stochastic formalism are naturally incorporated into Bohm's causal dynamics. Summarizing, here we generalize the approach of Ref.[14] to include dissipative two-particle states, thus allowing us to study Bohmian trajectories for correlated quantum systems described either by pure or mixed states under dissipation at finite temperature.

## II. BOHM'S TRAJECTORIES FOR TWO-PARTICLES DENSITY MATRICES

In this section we derive Bohm's trajectories from Bell's beables to encompass two entangled noninteracting particles under independent reservoirs at finite temperature, whose master equation is

$$\begin{aligned} \frac{\partial \rho(t)}{\partial t} = & -\frac{i}{\hbar} [H, \rho(t)] + \sum_{\alpha} \frac{\gamma_{\alpha}}{2} \left[ (\bar{n}_{\alpha} + 1) (2a_{\alpha} \rho a_{\alpha}^{\dagger} - a_{\alpha}^{\dagger} a_{\alpha} \rho - \rho a_{\alpha}^{\dagger} a_{\alpha}) \right. \\ & \left. + \bar{n}_{\alpha} (2a_{\alpha}^{\dagger} \rho a_{\alpha} - a_{\alpha} a_{\alpha}^{\dagger} \rho - \rho a_{\alpha} a_{\alpha}^{\dagger}) \right], \end{aligned} \quad (1)$$

with  $a_{\alpha}$  and  $a_{\alpha}^{\dagger}$  being the usual annihilation and creation operators in Fock spaces,  $\gamma_{\alpha}$  is the corresponding dissipative rate with reservoir average thermal photon number  $\bar{n}_{\alpha}$ , and  $\alpha = \{1, 2\}$  refers to each particle of the system and its corresponding

reservoir. This master equation generates the probability density  $P_{n_1 n_2}(t) = \langle \varphi_{n_1}, \chi_{n_2} | \rho(t) | \varphi_{n_1}, \chi_{n_2} \rangle$  such that

$$\hbar \frac{\partial P_{n_1 n_2}}{\partial t} = \langle \varphi_{n_1}, \chi_{n_2} | \frac{\partial \rho(t)}{\partial t} | \varphi_{n_1}, \chi_{n_2} \rangle \equiv \sum_{m_1, m_2} J_{n_1 n_2 m_1 m_2}. \quad (2)$$

Now, using the Eq. (1) and completeness relations we obtain

$$\begin{aligned} J_{n_1 n_2 m_1 m_2} &= 2\text{Im} \left\{ \langle \varphi_{n_1}, \chi_{n_2} | H | \varphi_{m_1}, \chi_{m_2} \rangle \langle \varphi_{m_1}, \chi_{m_2} | \rho(t) | \varphi_{n_1}, \chi_{n_2} \rangle \right\} \\ &+ \frac{\hbar}{2} \sum_{k_1, k_2, \alpha} \gamma_\alpha (\bar{n}_\alpha + 1) \left\{ 2 \langle \varphi_{n_1}, \chi_{n_2} | a_\alpha | \varphi_{m_1}, \chi_{m_2} \rangle \langle \varphi_{m_1}, \chi_{m_2} | \rho(t) | \varphi_{k_1}, \chi_{k_2} \rangle \langle \varphi_{k_1}, \chi_{k_2} | a_\alpha^\dagger | \varphi_{n_1}, \chi_{n_2} \rangle \right. \\ &- \langle \varphi_{n_1}, \chi_{n_2} | a_\alpha^\dagger | \varphi_{m_1}, \chi_{m_2} \rangle \langle \varphi_{m_1}, \chi_{m_2} | a_\alpha | \varphi_{k_1}, \chi_{k_2} \rangle \langle \varphi_{k_1}, \chi_{k_2} | \rho(t) | \varphi_{n_1}, \chi_{n_2} \rangle \\ &- \langle \varphi_{n_1}, \chi_{n_2} | \rho(t) | \varphi_{m_1}, \chi_{m_2} \rangle \langle \varphi_{m_1}, \chi_{m_2} | a_\alpha^\dagger | \varphi_{k_1}, \chi_{k_2} \rangle \langle \varphi_{k_1}, \chi_{k_2} | a_\alpha | \varphi_{n_1}, \chi_{n_2} \rangle \left. \right\} \\ &+ \frac{\hbar}{2} \sum_{k_1, k_2, \alpha} \gamma_\alpha \bar{n}_\alpha \left\{ 2 \langle \varphi_{n_1}, \chi_{n_2} | a_\alpha^\dagger | \varphi_{m_1}, \chi_{m_2} \rangle \langle \varphi_{m_1}, \chi_{m_2} | \rho(t) | \varphi_{k_1}, \chi_{k_2} \rangle \langle \varphi_{k_1}, \chi_{k_2} | a_\alpha | \varphi_{n_1}, \chi_{n_2} \rangle \right. \\ &- \langle \varphi_{n_1}, \chi_{n_2} | a_\alpha | \varphi_{m_1}, \chi_{m_2} \rangle \langle \varphi_{m_1}, \chi_{m_2} | a_\alpha^\dagger | \varphi_{k_1}, \chi_{k_2} \rangle \langle \varphi_{k_1}, \chi_{k_2} | \rho(t) | \varphi_{n_1}, \chi_{n_2} \rangle \\ &- \langle \varphi_{n_1}, \chi_{n_2} | \rho(t) | \varphi_{m_1}, \chi_{m_2} \rangle \langle \varphi_{m_1}, \chi_{m_2} | a_\alpha | \varphi_{k_1}, \chi_{k_2} \rangle \langle \varphi_{k_1}, \chi_{k_2} | a_\alpha^\dagger | \varphi_{n_1}, \chi_{n_2} \rangle \left. \right\}. \end{aligned} \quad (3)$$

As the classical counterpart to the continuity Eq. (2) we write the following master equation for two particles

$$\frac{\partial P_{n_1 n_2}}{\partial t} = \sum_{m_1, m_2} (T_{n_1 n_2 m_1 m_2} P_{m_1 m_2} - T_{m_1 m_2 n_1 n_2} P_{n_1 n_2}), \quad (4)$$

where  $T_{n_1 n_2 m_1 m_2} dt$  is the transition probability governing jumps from states  $|\varphi_{n_1}\rangle$  and  $|\chi_{m_1}\rangle$  to  $|\varphi_{n_2}\rangle$  and  $|\chi_{m_2}\rangle$ , respectively. The quantum and stochastic formalism meet a common ground through the mixed quantum-classical equation

$$\frac{J_{n_1 n_2 m_1 m_2}}{\hbar} = T_{n_1 n_2 m_1 m_2} P_{m_1 m_2} - T_{m_1 m_2 n_1 n_2} P_{n_1 n_2} \quad (5)$$

which admits the particular simplified solution

$$T_{n_1 n_2 m_1 m_2} = \begin{cases} \frac{J_{n_1 n_2 m_1 m_2}}{\hbar P_{m_1 m_2}} & , \quad J_{n_1 n_2 m_1 m_2} \geq 0 \\ 0 & , \quad J_{n_1 n_2 m_1 m_2} \leq 0 \end{cases}. \quad (6)$$

Next, we assume that the entangled systems are non-interacting harmonic oscillators of frequencies  $\omega_\alpha$  and masses  $M_\alpha$ , each modeled by the Hamiltonian

$$H^{(\alpha)} = \frac{p_\alpha^2}{2M_\alpha} + \frac{M_\alpha \omega_\alpha^2}{2} x_\alpha^2, \quad (7)$$

$p_\alpha$  being the canonically conjugate momentum to the coordinate variable  $x_\alpha$ , and the total Hamiltonian is  $H = H^{(1)} + H^{(2)}$ . Additionally, we consider both entangled systems to be described by the arbitrary general mixed state

$$\rho(t) = \sum_u P_u |\psi^u(t)\rangle \langle \psi^u(t)|, \quad (8)$$

with  $\langle \varphi_{n_1}, \chi_{n_2} | \psi^u(t) \rangle \equiv \psi_{n_1, n_2}^u$ .

In Vink's extension of Bell's beables [13], where all the degrees of freedom must be discrete and finite, the position is restricted to sites of a lattice which, in the one-dimensional case, becomes  $x_{n_\alpha} = n_\alpha \varepsilon$ ,  $n_\alpha$  being integers and  $\varepsilon$  is the lattice distance. To extend Vink's approach to two particles (the continuous limit is recovered taking  $\varepsilon \rightarrow 0$ ) we must *i*) write the smooth wave functions in the coordinate representations as  $(|\varphi_{n_\alpha}\rangle = |x_{n_\alpha}\rangle)$   $\psi_{n_1, n_2}^u = R_{n_1, n_2}^u \exp[\frac{i}{\hbar} S_{n_1, n_2}^u]$ , where  $\psi_{n_1, n_2}^u \equiv \psi^u(x_{n_1}, x_{n_2}, t)$ ,  $R_{n_1, n_2}^u \equiv R^u(x_{n_1}, x_{n_2}, t)$ , and  $S_{n_1, n_2}^u \equiv S^u(x_{n_1}, x_{n_2}, t)$ ; *ii*) expand  $\psi^u$  to first order in  $\varepsilon$ , *i.e.*,

$$\psi_{n_1 \pm 1, n_2}^u = \psi_{n_1, n_2}^u \pm \varepsilon \Delta_1 \psi_{n_1, n_2}^u \quad (9)$$

$$\psi_{n_1, n_2 \pm 1}^u = \psi_{n_1, n_2}^u \pm \varepsilon \Delta_2 \psi_{n_1, n_2}^u \quad (10)$$

with

$$\Delta_\alpha \psi_{n_1, n_2}^u = \left[ \Delta_\alpha R_{n_1, n_2}^u + \frac{i}{\hbar} R_{n_1, n_2}^u \Delta_\alpha S_{n_1, n_2}^u \right] \exp \left( \frac{i}{\hbar} S_{n_1, n_2}^u \right); \quad (11)$$

iii) substitute  $a_{n_\alpha m_\alpha} = [M_\alpha \omega_\alpha x_{m_\alpha} \delta_{n_\alpha, m_\alpha} + \hbar(\delta_{n_\alpha+1, m_\alpha} - \delta_{n_\alpha, m_\alpha})/\varepsilon]/(2\hbar M_\alpha \omega_\alpha)^{1/2}$  in the transition matrix given by  $J_{n_1 n_2 m_1 m_2}/\hbar$ , neglect terms of order  $O(\varepsilon^2)$  and higher (taking  $\varepsilon_\alpha = \varepsilon$  for simplicity); iv) take the limit  $\varepsilon \rightarrow 0$  with  $x_{m_2 \pm 1} = \varepsilon m_2 \pm \varepsilon$ . After a straightforward but length calculations we obtain  $(\alpha, \beta = 1, 2)$ :

$$\frac{J_{n_1 n_2 m_1 m_2}}{\hbar} = \sum_u P_u(t) R_u^2 \sum_{(\alpha \neq \beta)} \left[ \frac{1}{M_\alpha} \Delta_\alpha S_u - \frac{\hbar \gamma_\alpha (2\bar{n}_\alpha + 1)}{2M_\alpha \omega_\alpha} \frac{1}{R_u} \Delta_\alpha R_u \right] (\delta_{m_\alpha+1, n_\alpha} - \delta_{m_\alpha-1, n_\alpha}) \delta_{m_\beta, n_\beta}. \quad (12)$$

Next, by defining

$$x_\alpha(t + dt) \simeq x_\alpha(t) + \varepsilon[\langle k - m \rangle \delta_{\alpha, 1} + \langle l - n \rangle \delta_{\alpha, 2}], \quad (13)$$

with  $\langle k - m \rangle = \sum_{mn} T_{mkn} (k - m) dt$  and  $\langle l - n \rangle = \sum_{mn} T_{mkn} (l - n) dt$ , such that for the forward movement  $n_1 - m_1 = 1$  i.e.,  $k > m$  ( $l > n$ ) for particle 1 (2), Eq. (13) results in

$$\begin{aligned} \frac{dx_\alpha(t)}{dt} &= \frac{1}{\sum_u P_u(R^u(x_1, x_2, t))^2} \\ &\times \sum_u \left\{ P_u(R^u(x_1, x_2, t))^2 \left[ \frac{1}{M_\alpha} \frac{\partial S^u(x_1, x_2, t)}{\partial x_\alpha} + \frac{\hbar \gamma_\alpha (2\bar{n}_\alpha + 1)}{2M_\alpha \omega_\alpha} \left( \frac{1}{R^u(x_1, x_2, t)} \frac{\partial R^u(x_1, x_2, t)}{\partial x_\alpha} \right) \right] \right\}. \end{aligned}$$

The above equation can be rewritten more compactly in the following way, using  $\rho(x_1, x_2, x'_1, x'_2, t) = \langle x_1, x_2 | \rho(t) | x'_1, x'_2 \rangle$ , as

$$\frac{dx_\alpha(t)}{dt} = \frac{\hbar}{M_\alpha} \left[ \frac{\text{Im}[\partial_{x_\alpha} \rho(x_1, x_2, x'_1, x'_2, t)]}{\rho(x_1, x_2, x'_1, x'_2, t)} \right]_{x_1=x'_1, x_2=x'_2} + \frac{\hbar \gamma_\alpha (2\bar{n}_\alpha + 1)}{2M_\alpha \omega_\alpha} \left[ \frac{\text{Re}[\partial_{x_\alpha} \rho(x_1, x_2, x'_1, x'_2, t)]}{\rho(x_1, x_2, x'_1, x'_2, t)} \right]_{x_1=x'_1, x_2=x'_2}. \quad (14)$$

We note that our main result, Eq. (14), generalizes that one obtained by a different approach in Ref.[15] for density matrix of individual systems without dissipation ( $\gamma_\alpha = 0$ ). In the following, we shall use Eq. (14) to calculate trajectories of quantum particles when the entanglement either is present or absent in the joint state.

### III. GENERALIZED WERNER STATES AND MOTION EQUATIONS

With the Bohmian equation of motion for two particles in hands, Eq. (14), we next assume that the entangled state is prepared in the generalized Werner state

$$\rho = \epsilon |\psi^\pm\rangle_{12} \langle \psi^\pm| + \frac{1-\epsilon}{4} \mathbb{I}, \quad (15)$$

where  $\mathbb{I} \equiv \mathbb{I}_1 \otimes \mathbb{I}_2$  stands for the identity operator and

$$|\psi^\pm\rangle_{12} = a|00\rangle \pm b|11\rangle, \quad (16)$$

with  $a$  and  $b$  being complex constants ( $|a|^2 + |b|^2 = 1$ ).

To compute Bohmian trajectories we must find the solution  $\rho(t)$  in the presence of losses due to a thermal reservoir. We then will specialize to the case of losses at zero temperature, using the method of *phenomenological operator approach*, as developed in Ref.[19], where we define ( $\gamma_\alpha = \gamma$ )

$$\begin{aligned} |0\rangle|0\rangle_R &\rightarrow |0\rangle|0\rangle_R \\ |1\rangle|0\rangle_R &\rightarrow e^{-\frac{\gamma}{2}t} |1\rangle|0\rangle_R + \sqrt{1 - e^{-\gamma t}} |0\rangle|1\rangle_R. \end{aligned}$$

If we now take into account that the Werner-like state is uncoupled from the reservoir at  $t = 0$ , then using the rules given above we can write, after tracing out the reservoir variables,

$$\begin{aligned}
\rho(t) = & \left\{ \left[ \epsilon a^2 + \frac{1-\epsilon}{4} \right] + \left[ \epsilon b^2 + \frac{1-\epsilon}{4} \right] (1 - e^{-\gamma t})^2 + \frac{1-\epsilon}{2} (1 - e^{-\gamma t}) \right\} |00\rangle \langle 00| \\
& + \left[ \epsilon b^2 + \frac{1-\epsilon}{4} \right] e^{-2\gamma t} |11\rangle \langle 11| \\
& + e^{-\gamma t} \left\{ \left[ \epsilon b^2 + \frac{1-\epsilon}{4} \right] (1 - e^{-\gamma t}) + \frac{1-\epsilon}{4} \right\} (|10\rangle \langle 10| + |01\rangle \langle 01|) \\
& \pm \epsilon e^{-\gamma t} [ab^* e^{2i\omega t} |00\rangle \langle 11| + ba^* e^{-2i\omega t} |11\rangle \langle 00|],
\end{aligned} \tag{17}$$

where we have chosen  $\omega_\alpha = \omega$ . We now assume that both entangled particles, originally represented in abstract Fock spaces, are harmonic oscillators within the subspace  $\{0, 1\}$  of the ground and first excited states. This assumption enables us to analyze, through both particles' trajectories, how entanglement dynamics affects Bohmian trajectories. Considering the scaled dimensionless variables  $\tilde{x}_\alpha = \sqrt{\omega/\hbar} x_\alpha$ , we write the state (17) in the coordinate representation to obtain  $\rho(x_1, x_2, x'_1, x'_2, t)$  as

$$\begin{aligned}
\rho(x_1, x_2, x'_1, x'_2, t) = & \left( \frac{\omega}{\pi\hbar} \right) \left\{ \left[ \epsilon a^2 + \frac{1-\epsilon}{4} \right] + \left[ \epsilon b^2 + \frac{1-\epsilon}{4} \right] (1 - e^{-\gamma t})^2 + \frac{1-\epsilon}{2} (1 - e^{-\gamma t}) \right. \\
& + 4 \left[ \epsilon b^2 + \frac{1-\epsilon}{4} \right] e^{-2\gamma t} \tilde{x}_1 \tilde{x}_1' \tilde{x}_2 \tilde{x}_2' \\
& + 2e^{-\gamma t} \left\{ \left[ \epsilon b^2 + \frac{1-\epsilon}{4} \right] (1 - e^{-\gamma t}) + \frac{1-\epsilon}{4} \right\} [\tilde{x}_1 \tilde{x}_1' + \tilde{x}_2 \tilde{x}_2'] \\
& \left. \pm 2\epsilon e^{-\gamma t} [ab^* e^{2i\omega t} \tilde{x}_1' \tilde{x}_2' + ba^* e^{-2i\omega t} \tilde{x}_1 \tilde{x}_2] \right\} \times e^{-\frac{1}{2}[\tilde{x}_1^2 + (\tilde{x}_1')^2 + \tilde{x}_2^2 + (\tilde{x}_2')^2]}.
\end{aligned} \tag{18}$$

Using Eq. (14) we obtain, after a straightforward calculation

$$\frac{d\tilde{x}_1}{dt} = \frac{\gamma}{\mathcal{G}(\tilde{x}_1, \tilde{x}_2; t)} \left[ \mp \frac{2\omega}{\gamma} A(t) \sin(2\omega t) \tilde{x}_2 + B(t) \tilde{x}_1 \tilde{x}_2^2 + C(t) \tilde{x}_1 \pm A(t) \cos(2\omega t) \tilde{x}_2 \right] - \frac{\gamma \tilde{x}_1}{2}, \tag{19}$$

$$\frac{d\tilde{x}_2}{dt} = \frac{\gamma}{\mathcal{G}(\tilde{x}_1, \tilde{x}_2; t)} \left[ \mp \frac{2\omega}{\gamma} A(t) \sin(2\omega t) \tilde{x}_1 + B(t) \tilde{x}_1^2 \tilde{x}_2 + C(t) \tilde{x}_2 \pm A(t) \cos(2\omega t) \tilde{x}_1 \right] - \frac{\gamma \tilde{x}_2}{2}, \tag{20}$$

where

$$\begin{aligned}
\mathcal{G}(\tilde{x}_1, \tilde{x}_2; t) = & \left[ \epsilon a^2 + \frac{1-\epsilon}{4} \right] + \left[ \epsilon b^2 + \frac{1-\epsilon}{4} \right] (1 - e^{-\gamma t})^2 \\
& + \frac{1-\epsilon}{2} (1 - e^{-\gamma t}) + 4 \left[ \epsilon b^2 + \frac{1-\epsilon}{4} \right] e^{-2\gamma t} \tilde{x}_1^2 \tilde{x}_2^2 \\
& + 2e^{-\gamma t} \left\{ \left[ \epsilon b^2 + \frac{1-\epsilon}{4} \right] (1 - e^{-\gamma t}) + \frac{1-\epsilon}{4} \right\} [\tilde{x}_1^2 + \tilde{x}_2^2] \\
& \pm 4\epsilon ab \tilde{x}_1 \tilde{x}_2 e^{-\gamma t} \cos(2\omega t),
\end{aligned} \tag{21}$$

$$A(t) = \epsilon ab e^{-\gamma t}, \tag{22}$$

$$B(t) = 2 \left[ \epsilon b^2 + \frac{1-\epsilon}{4} \right] e^{-2\gamma t}, \tag{23}$$

$$C(t) = 2e^{-\gamma t} \left\{ \left[ \epsilon b^2 + \frac{1-\epsilon}{4} \right] (1 - e^{-\gamma t}) + \frac{1-\epsilon}{4} \right\}. \tag{24}$$

In the next section we will explore these solutions plotting the corresponding quantum trajectories for the generalized Werner state of Eq. (15) in regions occurring entanglement or separability.

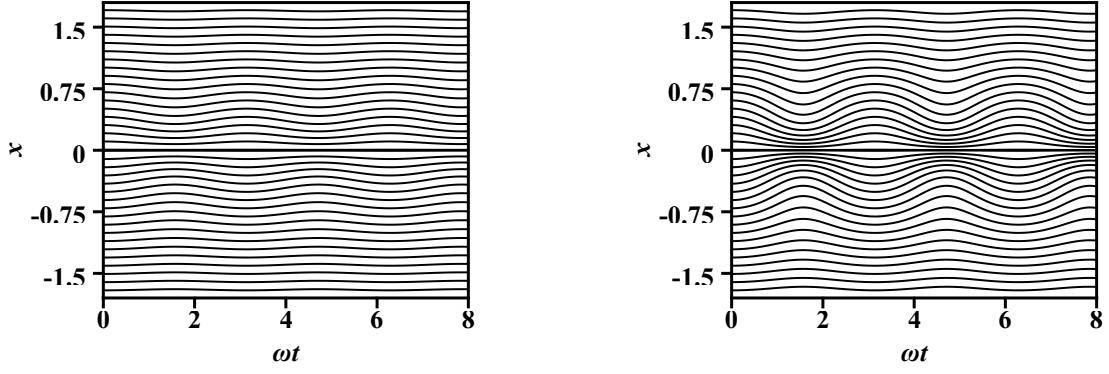


FIG. 1: Bohmian trajectories for the generalized Werner state of Eq. (15) with  $a = b$ . For separable states with (a)  $\epsilon = 0.1$  and (b)  $\epsilon = 1/3$ .

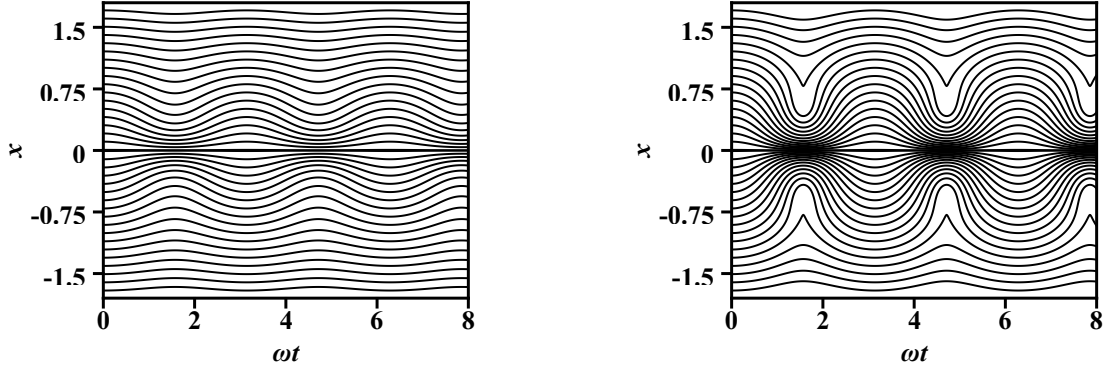


FIG. 2: Bohmian trajectories for the generalized Werner state of Eq. (15) with  $a = b$ . For entangled states with (a)  $\epsilon = 0.4$  and (b)  $\epsilon = 1$ .

#### IV. QUANTUM TRAJECTORIES FOR ENTANGLED STATES

In this section we present our results regarding Bohmian trajectories for both separable and entangled states using the generalized Werner state given in Eq. (15). To quantify the entanglement present in this state we can either use the concurrence [16] or the negativity [17, 18], thus we will use the concurrence as defined for two-qubit system:

$$C(\rho) = \max \left\{ 0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4} \right\}, \quad (25)$$

where  $\lambda_k$  are the eigenvalues of the matrix  $\tilde{\rho}_{12}(t) = \sigma_y^1 \sigma_y^2 \rho_{12}^*(t) \sigma_y^1 \sigma_y^2$  arranged in decreasing order.

As is well known, when disregarding losses and  $a = b$ , this state is separable for  $\epsilon = 1/3$ . In Figs. 1(a) and 1(b) we show the Bohmian trajectories for the generalized Werner state of Eq. (15) for  $a = b$  and  $\epsilon = 0.1, 1/3$ , corresponding to a separable state. In Figs. 2(a) and 2(b) we show the quantum trajectories when  $a = b$  and  $\epsilon = 0.4, 1$ , corresponding to an entangled state. From this sample of figures it can be seen that for separable state ( $\epsilon < 1/3$ ) the amplitude of oscillations of each trajectory is smooth and relatively small as compared with the corresponding trajectory (same initial condition) for entangled states ( $\epsilon > 1/3$ ), as advanced in Fig. 5(a) where we have plotted  $x(t)$  versus  $\epsilon$  for the same initial condition. From Figs. 2(a) and 2(b) we observe that, as the mixing parameter ( $\epsilon$ ) increases, there is a corresponding squeezing of the trajectories in regions where both particles approach each other. This squeeze of trajectories, which clearly increases from Fig. 2(a) to Fig. 2(b), together with the increasing oscillation amplitude, works as a signature of entanglement for the generalized Werner state studied here.

Using Eq. (25), we can readily check that disregarding losses and considering  $a = 0.2$ , this state is separable for  $\epsilon \leq 0.56$ . In Figs. 3(a) and 3(b) we show the corresponding trajectories for separable states, while in Figs. 4(a) and 4(b) we show the trajectories for entangled states. Note that the same behavior as that for  $a = b = 1/\sqrt{2}$  can be observed in Fig. 5(b) that consider  $a = 0.2$ : given the same initial conditions, quantum trajectories for separable states are smooth and oscillate less than those corresponding to entangled states. For a complete mixture ( $\epsilon = 0$ ), the trajectories are straight lines.

Now let us see what happens to Bohmian trajectories in the presence of losses. Again using Eq. (25), we can check that considering  $a = b$  and  $\epsilon = 0.4$  this state is separable at  $\gamma t = 0.15$ . Figs. 6(a,b) and 7(a,b) show the corresponding trajectories

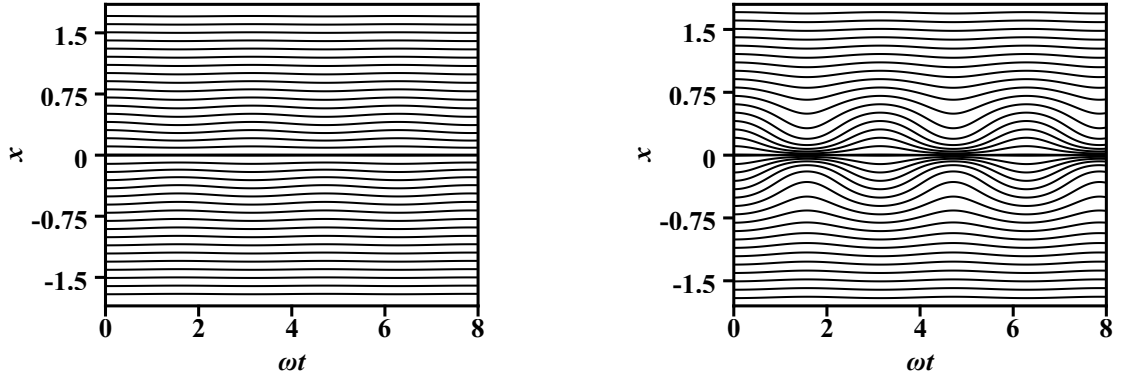


FIG. 3: Bohmian trajectories for the generalized Werner state of Eq. (15) with  $a = 0.2$ . Entanglement occurs for  $\epsilon \geq 0.56$ . (a)  $\epsilon = 0.1$  and (b)  $\epsilon = 0.56$ .

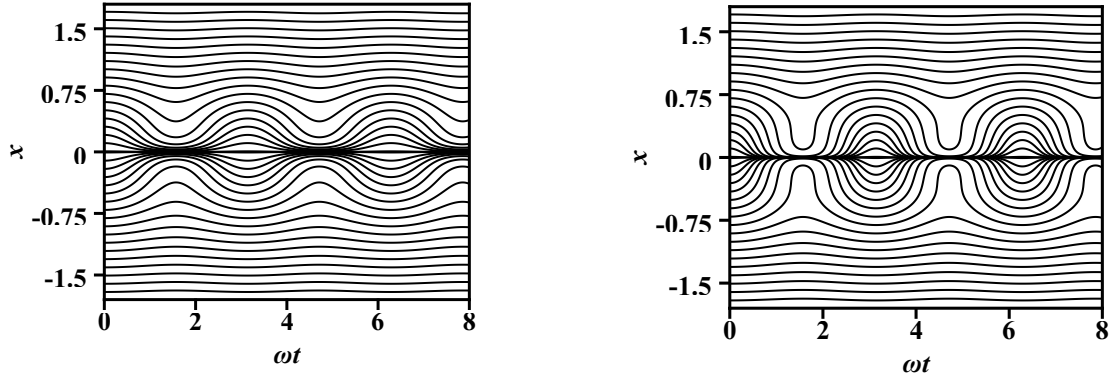


FIG. 4: Bohmian trajectories for the generalized Werner state of Eq. (15) with  $a = 0.2$ . Entanglement occurs for  $\epsilon \geq 0.56$ . (a)  $\epsilon = 0.7$  (b) and  $\epsilon = 1.0$ .

for damped states. As expected, the reservoir attenuates the oscillations, that turn to be more accentuated when the damping rate is larger.

Note that these results for damped states show the same behavior as that for undamped states: given the same initial conditions, quantum trajectories for separable states are smooth and the amplitude of oscillations is less than those corresponding to entangled states, with the amplitude of oscillations going to zero when  $\epsilon \rightarrow 0$ .

As a final remark, it is worth noting that Bohmian trajectories may provide a way towards detecting quantum separability of mixed quantum states. As a matter of fact, when the interpolator parameter  $\epsilon$  evolves from 0 to 1, the trajectories changes from straight lines to curves with steep slopes. However, around the regions where the trajectories of the particles come closer together—thus interfering to a greater extent—their curvatures become very smooth. By focusing our attention on these regions of maximum interference between the trajectories, which takes place in Figs. 2 and 4 for  $\omega t = (2n + 1)\pi/2$ ,  $n = 0, 1, 2, \dots$ , we thus observe that when the parameter  $\epsilon$  evolves from 0 to 1, the slopes of the trajectories starts from 0, seems to reach a maximum value and then decreases due to the strong interference between the particles paths. We might suspect that the maximum curvature takes for the value of  $\epsilon$  that gives the separability condition for the density matrix. However, a problem arise when we set out to compute the curvature of the trajectories (in the specified regions) as a function of the parameter  $\epsilon$ , since the curvatures of the trajectories are different for different initial positions  $\tilde{x}_\alpha$  of the particles.

## V. CONCLUSIONS

In this paper we have derived Bohmian trajectories for noninteracting bipartite states of damped harmonic oscillators under a thermal reservoir at finite temperature in a similar way to that of Vink's extension of Bell's beables[13]. As an application, we have calculated the trajectories for a generalized Werner state dissipating at zero temperature in regions where the two systems are either entangled or separable according to Wootters' concurrence. Our results indicated that individual trajectories for entangled

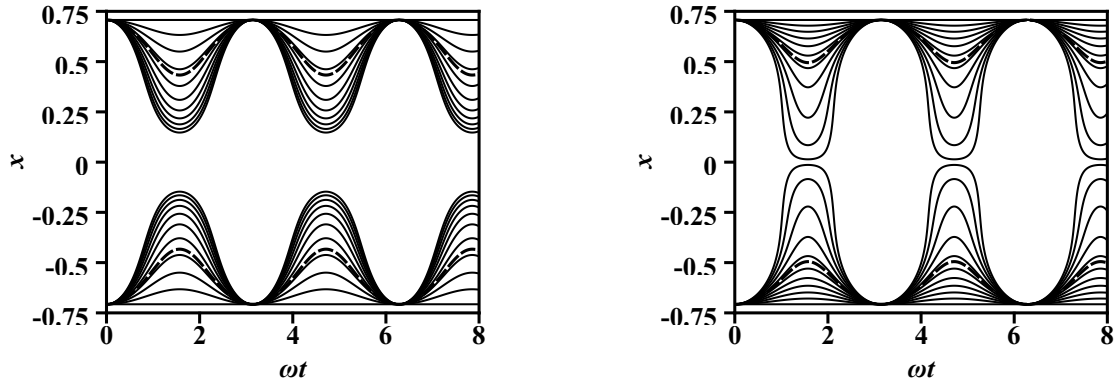


FIG. 5: Bohmian trajectories for (a)  $a = b$  and (b)  $a = 0.2$ , for the same initial conditions and several values of  $\epsilon$ . The constant trajectory is for  $\epsilon = 0$ , while the oscillation amplitude is maximum for  $\epsilon = 1.0$ . The trajectories for (a)  $\epsilon = 1/3$  and (b)  $\epsilon = 0.56$  are indicated by a dashed lines.

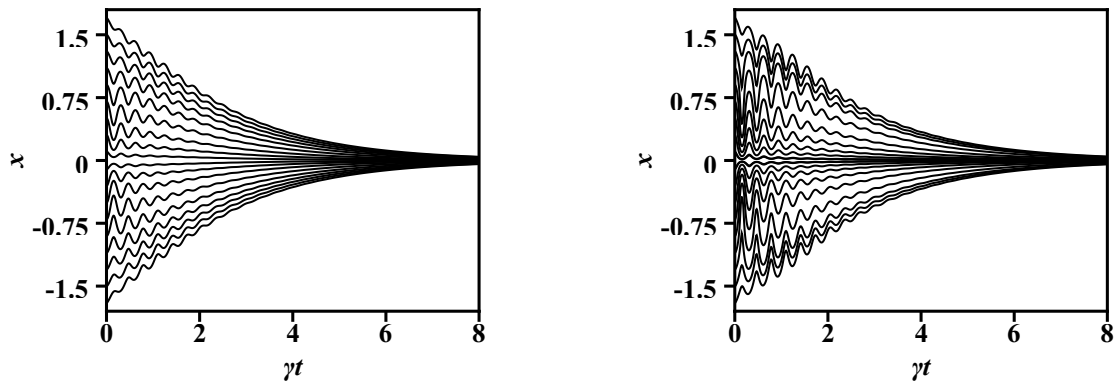


FIG. 6: Bohmian trajectories for the generalized Werner state of Eq. (15) with  $a = b$ ,  $\omega/\gamma = 10.0$  (a) for  $\epsilon = 0.4$ , where entanglement is suddenly lost at  $\gamma t = 0.15$ , and (b) for  $\epsilon = 1.0$ , where entanglement is lost at the asymptotic time.

states differ slightly in the amplitude of oscillation as compared with those corresponding (same initial conditions) trajectories for disentangled states. We note, however, that according to our simulations, this difference is not enough to characterize unambiguously separability or entanglement, which is a global property of the system. This is so because the trajectories change continuously when the state changes from separable to nonseparable. We hope these preliminary results can encourage future research towards an eventual link between separability and Bohmian mechanics.

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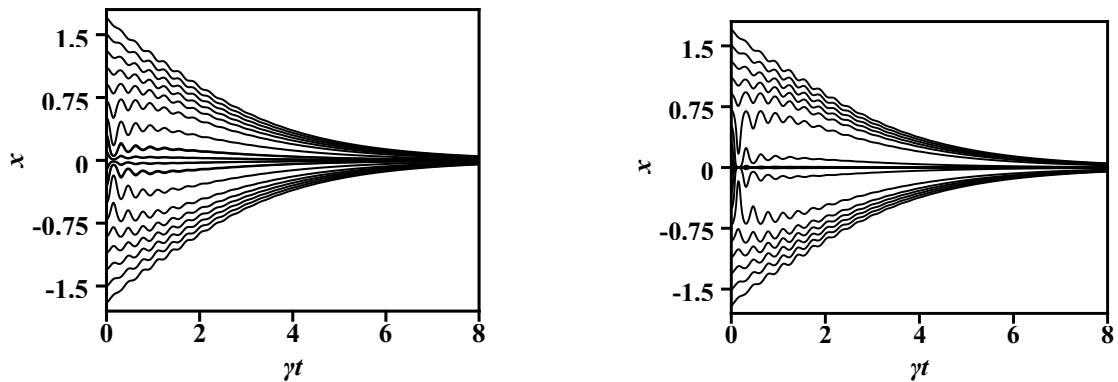


FIG. 7: Bohmian trajectories for the generalized Werner state of Eq. (15) with  $a = 0.2$ , and  $\omega/\gamma = 10.0$  for (a) for  $\epsilon = 0.7$ , where entanglement is suddenly lost at  $\gamma t = 0.026$  and (b)  $\epsilon = 1.0$ , where entanglement is lost at  $\gamma t = 0.23$ .